ON THE EQUATIONS OF MOTION FOR A HEAVY BODY WITH A FIXED POINT

(OB URAVNENIIAKH DVIZHENIIA TIAZHELOGO TVERDOGO TELA, IMEIUSHCHEGO NEPODVIZHNUIU TOCHKU)

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The three known integrals of the Euler equations describing the subject motion permit, in principle, the lowering of the order of the equations of motion from the sixth to the third, and by eliminating the independent variable, to the second order.

Many attempts were made to realize such order reduction. However, in a number of papers such efforts were incomplete (for example, the obtained system was of fourth order), or were subject to certain limitations (for example, requiring that the body inertia ellipsoid be an ellipsoid of rotation), or finally, these attempts required operations the results of which could not be expressed explicitely in the general case (for example, the solution of an algebraic equation of a general type of sufficiently high order).

In the present paper, the considered problem in the general case is reduced to two equations of first order each. These equations are of simplest form in the special rectangular system of coordinates the axes of which, generally speaking, do not coincide with the principal axes of the inertia ellipsoid for the body with a fixed point. At the same time the Chaplygin-Kowalewski equations are generalized. The restriction on the location of center of gravity of the body is removed.

1. Order reduction of the Euler equations. In the body-centered system of coordinates the time-variation of the vector **x**, the angular momentum of the body relative to its fixed point, and of the vector γ directed along the gravity force and having the magnitude equal to the product of the body weight and the distance between the fixed point and the center of gravity, is described by the Euler equations [1]

$$d\mathbf{x}/dt + \boldsymbol{\omega} \times \mathbf{x} = \mathbf{e} \times \boldsymbol{\gamma}, \qquad d\boldsymbol{\gamma}/dt + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0 \tag{1.1}$$

Here $\boldsymbol{\omega}$ is the angular velocity of the body, and $\mathbf{e}$ is the unit vector directed from the fixed point to the center of gravity of the body.

Let us reduce the order of equations (1.1), utilizing the known integrals

$$T - \mathbf{e}\cdot\boldsymbol{\gamma} = h, \qquad \mathbf{x}\cdot\boldsymbol{\gamma} = m, \qquad \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = \gamma^2 \tag{1.2}$$

Here $T = 1/2\ \boldsymbol{\omega} \times \mathbf{x}$ is the kinetic energy of the body, and h and m are integration constants. From the first equation in (1.1) and the first integral in (1.2) we find

$$\boldsymbol{\gamma} = (T - h)\ \mathbf{e} + (d\mathbf{x}/dt + \boldsymbol{\omega} \times \mathbf{x}) \times \mathbf{e} \tag{1.3}$$

Let us substitute (1.3) into the second and third integrals of (1.2)

$$(d\mathbf{x}/dt + \boldsymbol{\omega} \times \mathbf{x})\cdot(\mathbf{e} \times \mathbf{x}) + (T - h)(\mathbf{e}\cdot\mathbf{x}) = m \qquad (d\mathbf{x}/dt + \boldsymbol{\omega} \times \mathbf{x})^2 + (T - h)^2 = \gamma^2 \tag{1.4}$$

Also, the first equation in (1.1) yields

$$\mathbf{e}\,(d\mathbf{x}/dt + \boldsymbol{\omega} \times \mathbf{x}) = 0 \tag{1.5}$$

The three equations (1.4) and (1.5) define the components of the vector $\mathbf{x}$ but do not contain γ.

If the coordinate axes are directed along the principal axes of the inertia ellipsoid for the body with a fixed point, and the projections of the vectors $\boldsymbol{\omega}$, $\mathbf{x}$, γ, $\mathbf{e}$ are respectively denoted by p, q, r; Ap, Bq, Cr; γ_1, γ_2, γ_3 and e_1, e_2, e_3, then equations (1.1) and integrals (1.2) are expressed as

$$A\,dp/dt + (C - B)\,qr = e_2\gamma_3 - e_3\gamma_2, \qquad d\gamma_1/dt = r\gamma_2 - q\gamma_3 \quad (123),\ (pqr),\ (ABC) \tag{1.6}$$

$$\tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) - e_i\gamma_i = h, \qquad Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = m, \qquad \gamma_i\gamma_i = \gamma^2 \tag{1.7}$$

The omitted equations are obtainable from (1.6) by simultaneous cyclic permutation of indices and letters indicated in the parentheses. In (1.7) double indexing denotes summation from 1 to 3. Such shortened notation is also used in the sequel.

Using the principal axes, equations (1.4) and (1.5) are

$$d/dt\,(Ape_1 + Bqe_2 + Cre_3) + (C - B)\,e_1qr + (A - C)\,e_2rp + (B - A)\,e_3pq = 0$$

$$(Cre_2 - Bqe_3)\,[A dp/dt + (C - B)\,qr] + (Ape_3 - Cre_1)\,[B dq/dt + (A - C)\,rp] + \\ + (Bqe_1 - Ape_2)\,[C dr/dt + (B - A)\,pq] + \\ + (Ape_1 + Bqe_2 + Cre_3)\,[{}^1/_2\,(Ap^2 + Bq^2 + Cr^2) - h] = m \tag{1.8}$$

$$[A dp/dt + (C - B)\,qr]^2 + [B dq/dt + (A - C)\,rp]^2 + [C dr/dt + (B - A)\,pq]^2 + \\ + [{}^1/_2\,(Ap^2 + Bq^2 + Cr^2) - h]^2 = \gamma^2$$

and formulas (1.3) are

$$\gamma_1 = Be_3 dq/dt - Ce_2 dr/dt + [(A - B)\,e_2 q + (A - C)\,e_3 r]\,p + e_1\,[{}^1/_2\,(Ap^2 + \\ + Bq^2 + Cr^2) - h] \qquad (123),\ (pqr),\ (ABC)$$

It is more convenient to use the special rectangular system of coordinates. The first coordinate axis is directed through the center of gravity of the body, while the second and third axis will be directed so that the product yz will be missing in the expression for the kinetic energy of the body which is a quadratic form of the components of vector $\mathbf{x}$:

$$T = {}^1/_2\,(ax^2 + a_1y^2 + a_2z^2) + (b_1y + b_2z)\,x$$

(the constants a, a_1, a_2, b_1 and b_2 are determined from the mass distribution in the body). then (1, 0, 0) are the components of the unit vector $\mathbf{e}$

$$\omega_1 = ax + b_1y + b_2z, \qquad \omega_2 = a_1y + b_1x, \qquad \omega_3 = a_2z + b_2x$$

the components of the vector γ are denoted as previously by γ_1, γ_2 and γ_3.

Equations (1.1) and integrals (1.2) referred to the special axes are

$$dx/dt = (a_2 - a_1)\,yz + (b_2y - b_1z)\,x \tag{1.9}$$

$$dy/dt = (a - a_2)\,xz + (b_1y + b_2z)\,z - b_2x^2 - \gamma_3 \tag{1.10}$$
$$dz/dt = -(a - a_1)\,xy - (b_1y + b_2z)\,y + b_1x^2 + \gamma_2$$

$${}^1/_2\,(ax^2 + a_1y^2 + a_2z^2) + (b_1y + b_2z)\,x - \gamma_1 = h, \qquad x_i\gamma_i = m, \qquad \gamma_i\gamma_i = \gamma^2 \tag{1.11}$$

Equation (1.5) reduces to (1.9), while equations (1.4) become

$$y\,dz/dt - z\,dy/dt + (y^2 + z^2)(b_1y + b_2z) + \\ + x\,[(a - {}^1/_2a_1)\,y^2 + (a - {}^1/_2a_2)\,z^2] + {}^1/_2ax^3 - hx = m$$

$$[(a_2 - a)\,xz - (b_1y + b_2z)\,z + b_2x^2 + dy/dt]^2 + \tag{1.12}$$

$$+ [(a_1 - a)\, xy - (b_1 y + b_2 z)\, y + b_1 x^2 - dz \,/\, dt]^2 +$$
$$+ [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h]^2 = \gamma^2$$

The solutions for which x = const, are obtained from system (1.9) and (1.12) only in the cases of Lagrange [2], Hess [3], Bobylev [4] and Steklov [5], and in the case of body motion with a fixed axis. Disregarding these cases we choose x as an independent variable and reduce the problem to two equations of first order

$$[(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, (ydz \,/\, dx - zdy \,/\, dx) + (y^2 + z^2)\, (b_1 y + b_2 z) +$$
$$+ x\, [(a - {}^1/_2 a_1)\, y^2 + (a - {}^1/_2 a_2)\, z^2] + {}^1/_2 a x^3 - hx - m = 0$$

$$\{[(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, dy \,/\, dx + (a_2 - a)\, xz - (b_1 y + b_2 z)\, z + b_2 x^2\}^2 +$$
$$+ \{[(a_1 - a_2)\, yz + (b_1 z - b_2 y)\, x]\, dz \,/\, dx + (a_1 - a)\, xy - (b_1 y + b_2 z)\, y + b_1 x^2\}^2 +$$
$$+ [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h]^2 - \gamma^2 = 0 \qquad (1.13)$$

Equations (1.13) permit interchange of the quantities y and z for simultaneous interchange of the indices 1 and 2.

Finding the dependence of y and z on x from (1.13), the dependence of t on x is found from (1.9) by quadrature. The dependence of γ_1, γ_2 and γ_3, on x is given by the formulas

$$\gamma_1 = {}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h$$
$$\gamma_2 = (a - a_1)\, xy + (b_1 y + b_2 z)\, y - b_1 x^2 + [(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, dz \,/\, dx$$
$$\gamma_3 = (a - a_2)\, xz + (b_1 y + b_2 z)\, z - b_2 x^2 + [(a_1 - a_2)\, yz + (b_1 z - b_2 y)\, x]\, dy \,/\, dx$$

which were obtained from (1.10), (1.11) and (1.9).

The equations indicated by Hess [3] follow from (1.8). Analogously, we find from (1.13) that

$$(y^2 + z^2)\, [(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, dy \,/\, dx =$$
$$= (y^2 + z^2)\, [(a - a_2)\, xz + (b_1 y + b_2 z)\, z - b_2 x^2] +$$
$$+ xz\, [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h] - mz + yR$$

$$(y^2 + z^2)\, [(a_1 - a_2)\, yz + (b_1 z - b_2 y)\, x]\, dz \,/\, dx =$$
$$= (y^2 + z^2)\, [(a - a_1)\, xy + (b_1 y + b_2 z)\, y - b_1 x^2] +$$
$$+ xy\, [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h] - my - zR$$

where

$$R^2 = (y^2 + z^2)\, \gamma^2 + 2mx\, [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h] -$$
$$- (x^2 + y^2 + z^2)\, [{}^1/_2\, (ax^2 + a_1 y^2 + a_2 z^2) + (b_1 y + b_2 z)\, x - h]^2 - m^2$$

2. Equations of motion in oblique coordinates. Let us refer to the body a rectilinear oblique system of coordinates with the origin at the fixed point. The directed vectors $\mathfrak{z}_i$ (i = 1, 2, 3) along the axes of this system, generally speaking, have a different modulus and form the main coordinate basis.

For the vectors $\mathfrak{z}^k$ of the mutual basis we have the formulas $\epsilon_{ijk}\mathfrak{z}^k = \mathfrak{z}_i \times \mathfrak{z}_j$, where

$$\epsilon_{ijk} = \mathfrak{z}_i \cdot (\mathfrak{z}_j \times \mathfrak{z}_k) \tag{2.1}$$

ϵ_{ijk} is the Levi-Civita tensor [6].

Let us define the vector for angular rotation by the relationship [7]

$$\omega^k = \frac{1}{2}\epsilon_{i\cdot\cdot}^{\cdot jk}\omega_{\cdot j}^{i\cdot} \qquad \left(\omega_{i\cdot}^{\cdot j} = \frac{d\mathfrak{z}_i}{dt}\cdot\mathfrak{z}^j - \text{ is the angular velocity tensor}\right)$$

We have $\omega_{\cdot j}^{i\cdot} = \epsilon_{\cdot jk}^{i}\omega^k$, and differentiating the radius-vector $\mathbf{r} = r^i\mathfrak{z}_i$ of a point on the body, we find its velocity $v^j = \epsilon_{i\cdot k}^{\cdot j\cdot}r^i\omega^k$; thereafter we have the vector

$$\mathbf{x} = \int \mathbf{r}\times\mathbf{v}\,dm \quad \text{or} \quad x_i = A_{is}\omega^s \qquad \left(A_{is} = \epsilon_{jki}\epsilon_{l\ s}^{\ k}\int r^l r^j dm\right)$$

where A_{is} is the inertia tensor. Since det $|A_{is}| \neq 0$ then $\omega^s = a^{si}x_i$. The symmetric tensor a^{si} can be naturally called a gyration tensor. Let us substitute $\mathbf{x} = x_i\mathfrak{z}^i$, $\boldsymbol{\gamma} = \gamma^i\mathfrak{z}_i$, $\mathbf{e} = e^i\mathfrak{z}_i$; into equations

$$d\mathbf{x}/dt = \mathbf{e}\times\boldsymbol{\gamma}, \quad d\boldsymbol{\gamma}/dt = 0$$

characterizing the variation of the angular momentum of the body and the constancy of the vector γ in the fixed coordinate system.

We then obtain

$$dx_i/dt + \epsilon^j_{\ ik}a^{kl}x_l x_j = \epsilon_{kji}e^k\gamma^j \tag{2.2}$$

$$d\gamma^i/dt + \epsilon_{j\ k}^{\ i}a^{kl}x_l\gamma^j = 0 \tag{2.3}$$

The closed system of Euler equations is of sixth order. The following integrals are known

$$\tfrac{1}{2}a^{ij}x_i x_j - e_i\gamma^i = h, \qquad x_i\gamma^i = m, \qquad \gamma_i\gamma^i = \gamma^2 \tag{2.4}$$

In the following the easily verifiable relationship is used

$$\varepsilon^{imn}\varepsilon_{ikj}u_{\cdot\cdot m}^{\cdot\cdot\cdot}w_{\cdot\cdot}^{k\cdot} = g_{\cdot j}^{n\cdot}u_{\cdot\cdot s}^{\cdot\cdot}w_{\cdot\cdot}^{s\cdot} - u_{\cdot\cdot j}^{\cdot\cdot}w_{\cdot\cdot}^{n\cdot} \qquad (g_{\cdot j}^{n} = \partial^{n}\cdot\partial_{j} \text{ is a metric tensor}) \tag{2.5}$$

stemming from (2.1) and valid for any tensors $u_{\cdot\cdot m}^{\cdot\cdot}$ and $w_{\cdot\cdot}^{k\cdot}$.

Multiplying (2.2) by $\varepsilon^{imn}e_m$, taking into account (2.5) and the first integral in (2.4), we obtain

$$\gamma^n = \varepsilon^{imn}e_m\,(dx_i\,/\,dt + \varepsilon^{j}_{\ ik}a^{kl}x_lx_j) + e^n\,(^1/_2a^{is}x_ix_s - h) \tag{2.6}$$

while by substituting (2.6) into the second and third integral in (2.4) we find

$$\begin{gathered}\varepsilon^{imn}e_mx_n\,(dx_i\,/\,dt + \varepsilon^{j}_{\ ik}a^{kl}x_lx_j) + e^nx_n\,(^1/_2a^{is}x_ix_s - h) = m\\ g^{is}\,(dx_i\,/\,dt + \varepsilon^{j}_{\ ik}a^{kl}x_lx_j)\,(dx_s\,/\,dt + \varepsilon^{p}_{\ sm}a^{mn}x_nx_p) + (^1/_2a^{sp}x_sx_p - h) = \gamma^2\end{gathered} \tag{2.7}$$

Along with the equation

$$e^i\,(dx_i\,/\,dt + \varepsilon^{j}_{\ ik}a^{kl}x_lx_j) = 0 \tag{2.8}$$

obtainable from (2.2), equations (2.7) constitute a system of third order.

Substituting now (2.6) into (2.3) we obtain

$$\begin{gathered}\varepsilon^{jki}e_kd^2x_i\,/\,dt^2 + (2g^{il}a^{ks} + g^{is}a^{kl} - g^{ks}a^{il} - g^{kl}a^{is})\,e_kx_sdx_l\,/\,dt +\\ + \varepsilon_{j\ n}^{\ i}a^{nm}\,(^1/_2a^{sl}g^{kj} + g^{js}a^{kl})\,e_kx_mx_sx_l - h\varepsilon_{j\ k}^{\ i}a^{kl}e^jx_l = 0\end{gathered} \tag{2.9}$$

The determinant of $|\varepsilon^{jki}e_k|$ is equal to zero and, consequently, a certain linear combination of equations (2.9) does not contain second derivatives. Multiplying (2.9) by e_i, we obtain equation (2.8) which along with the two equations in (2.9), constitutes a system of fifth order.

3. Reduction of the equations of motion to second order. The equations (2.7) and (2.8) are written in an arbitrary coordinate system. Now we will direct the first coordinate axis through the center of gravity of the body. Also $e^1 = e$, $e^2 = e^3 = 0$, $e_i = g_{i1}e$, and equation (2.8) is given by

$$dx_1\,/\,dt + \sqrt{g}\,(g^{3j}a^{2l} - g^{2j}a^{3l})\,x_jx_l = 0 \tag{3.1}$$

where $g = \det|g_{ij}|$. Disregarding the known solutions noted in Section 1 for which $x_1 = \text{const}$, we eliminate t from (2.7) with the aid of (3.1), as well as with the use of the relationship $\in^{j}{}_{1k} a^{kl} = \sqrt{g}\ (g^{3j} a^{2l} - g^{2j} a^{3l})$ (123) already used in deriving equation (3.1). We obtain

$$[(g_{31}x_1 - g_{11}x_3)\, dx_2 / dx_1 - (g_{21}\, x_1 - g_{11}x_2)\, dx_3 / dx_1]\, (g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l + \\ + [(g_{31}x_1 - g_{11}x_3)\, (g^{1j}a^{3l} - g^{3j}a^{1l}) + (g_{21}x_1 - g_{11}x_2)\, (g^{1j}a^{2l} - g^{2j}a^{1l})]\, x_j x_l + \\ + x_1 (^1/_2 a^{jl} x_j x_l - h) - m/e = 0 \qquad (3.2)$$

$$g^{22}\, [(g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l dx_2 / dx_1 + (g^{1j}a^{3l} - g^{3j}a^{1l})\, x_j x_l]^2 + \\ + g^{33}\, [(g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l dx_3 / dx_1 + (g^{2j}a^{1l} - g^{1j}a^{2l})\, x_j x_l]^2 + \\ + 2g^{23}\, [(g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l dx_2 / dx_1 + \\ + (g^{1j}a^{3l} - g^{3i}a^{1l})\, x_j x_l]\, [(g^{2i}a^{3k} - g^{3i}a^{2k})\, x_i x_k dx_3 / dx_1 + (g^{2i}a^{1k} - g^{1i}a^{2k})\, x_i x_k] + \\ + (^1/_2\, a^{ik} x_i x_k - h)^2 - \gamma^2 = 0$$

Thus, with the exception of the cases x_1 = const, the problem of the motion of a heavy rigid body having a fixed point is reduced to two ordinary differential equations of first order each.

Having determined from these equations the dependence of x_2 and x_3 on x_1, we establish by quadrature the connection between x_1 and t from (3.1).

The quantities γ^i are then found without integration from formulas (2.7) which in this case become

$$\frac{\gamma^1}{e} = \left(g_{31}\frac{dx_2}{dx_1} - g_{21}\frac{dx_3}{dx_1}\right)(g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l + \left(\frac{1}{2}\, a^{jl} + a_1{}^l g^{1j}\right) x_j x_l - h \\ \gamma^2 = g_{11} e \left[(g^{2j}a^{3l} - g^{3j}a^{2l})\, x_j x_l \frac{dx_3}{dx_1} + (g^{2j}a^{1l} - g^{1j}a^{2l})\, x_j x_l\right] \ (23) \qquad (3.3)$$

In passing to the special rectangular system of coordinates the equations (3.1), (3.2) and (3.3) reduce to (1.10), (1.13) and (1.14). One of the equations in (2.9) coincides with (1.10), while the remaining two equations become after elimination of the variable t

$$[(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, \frac{d}{dx}\, [(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x]\, \frac{dy}{dx} - \\ - [(a_2 - a_1)\, yz + (b_2 y - b_1 z)\, x] \left\{[(2a - a_2)\, x + 3b_2 z + 2b_1 y]\, \frac{dz}{dx} + b_1 z\frac{dy}{dx}\right\} + \\ + (^1/_2 a_1{}^2 - b_1{}^2)\, y^3 - 2b_1 b_2 y^2 z + [(a - a_2)\, (a_1 - a_2) + {}^1/_2 a_1 a_2 - b_2{}^2]\, yz^2 + \\ + {}^1/_2 b_1\, (5a_1 - 4a)\, xy^2 + 3b_2\, (a_2 - a_1)\, xyz + {}^1/_2 b_1\, (2a - a_2)\, xz^2 + \\ + \{x^2\, [2\, (b_1{}^2 + b_2{}^2) - {}^1/_2 a\, (2a - 3a_1)] - ha_1\}\, y + {}^3/_2 a b_1 x^3 - h b_1 x = 0 \quad (12),\ (yz) \qquad (3.4)$$

Evidently, (1.13) are the first integrals of the equations (3.4). In particular, if the body center of gravity is on one of the principal axes of the gyrational ellipsoid, then (3.4) yield the equations of Chaplygin-Kowalewski. Indeed, letting $b_1 = b_2 = 0$, $a = 1/A$, $a_1 = 1/B$, $a_2 = 1/C$, $x = Ap$, $y = Bq$, $z = Cr$ we obtain from (3.4)

$$\frac{B}{A}(B-C)^2\left(r^2\frac{d^2q^2}{dp^2}+\frac{1}{2}\frac{dr^2}{dp}\frac{dq^2}{dp}\right)-(B-C)(2C-A)\,p\frac{dr^2}{dp}+$$

$$+Aq^2+[2(C-A)(C-B)+AC]\,r^2+A(3A-2B)\,p^2-2hA=0 \quad (qr),\ (BC)$$

Introducing now the new variables

$$\sigma=\frac{B-C}{A}q^2,\qquad \tau=\frac{B-C}{A}r^2$$

we obtain the Kowalewski equations. The Chaplygin equations are obtained with

$$q^2=\frac{A}{B}\frac{C-A}{B-C}p^2-\frac{2}{B}\tau,\qquad r^2=\frac{A}{C}\frac{A-B}{B-C}p^2+\frac{2}{C}\sigma$$

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